Defects and the Convergence of Padé Approximants *

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Abstract. A selective survey is given of convergence results for sequences of Padé approximants. Various approaches for dealing with the convergence problems due to "defects" are discussed. Attention is drawn to the close relationship between analyticity properties of a function and the "smoothness" of its Taylor series coefficients. A new theorem on the convergence of horizontal sequences of Padé approximants to functions in the Baker-Gammel-Wills conjecture function class is presented.

Since the beginning of the age of computers, which made the quick and easy computation of Padé approximants [1] possible, people have noticed that while they generally converge in an excellent manner, these approximants are bedeviled by "defects." By a defect I mean a spurious pole and zero very close together. An early cited example [2] is for the function,

$$f(z) = \left[\int_0^z \left(\frac{\sin y}{y^3} - \frac{\cos y}{y^2}\right)^2 dy\right]^2$$

where the [5/5] Padé approximant has the poles and zeros

$$\text{Root of the numerator} = 2.8852000$$
$$\text{Root of the denominator} = 2.8851989$$

The first thought was that these were the same, except for numerical error, however not so!!!

There are three approaches to this problem. The first is analytic proof for special classes of functions. For example (i) series of Stieltjes, and (ii) Polya frequency series. Also there are a number of special cases where the exact Padé approximants can be worked out, *e.g.*, the continued fraction of Gauss. The form of the functions in those classes is:

$$\text{(i)} \quad f(z) = \int_0^\infty \frac{d\varphi(u)}{1+uz} \quad \text{where } d\varphi(u) \geq 0,$$
$$\text{(ii)} \quad f(z) = a_0 e^{\gamma z} \frac{\prod_{j=1}^\infty (1+\alpha_j z)}{\prod_{j=1}^\infty (1-\beta_j z)} \quad \text{where } a_0 > 0,$$

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$$\gamma, \alpha_j, \beta_j \geq 0, \; \sum_j (\alpha_j + \beta_j) < \infty.$$

In these cases the spurious poles are under control. That is to say they don't continually intrude into regions of convergence. The convergence in these cases is well understood.

The second approach is to look for a type of convergence other than pointwise. As on the previous page, it was noted numerically that the region of disruption of these defects was always small.

Here and there, in order to meet the needs of the presentation, I have reformulated, condensed, extracted, and/or generalized the theorems, *etc.*, quoted. I have however, I think, retained their essence.

An idea towards this approach is Cartan's lemma [3]. The point is to bound the denominator of the Padé approximant away from zero in most of the complex plane.

CARTAN'S LEMMA. *If* $P(z) = \prod_{\nu=1}^{n}(z - z_\nu)$, *then for any* $H > 0$,

$$|P(z)| > \left(\frac{H}{e^{1/\alpha}}\right)^n$$

holds outside at most n *circles whose radii satisfy,*

$$\sum_{i=1}^{n} r_i^\alpha \leq (2H)^\alpha.$$

If we choose $\alpha = 2$ then this lemma gives a result very much like the ordinary measure of the set where the polynomial is less that $(H/\sqrt{e})^n$. For general α this measure of set size is the Hausdorff measure of dimension α. For $\alpha \approx 0$ the Hausdorff measure is similar to, but distinct from the measure called capacity.

DEFINITION OF CAPACITY. *Let* $T_m(z)$ *be the Tschebycheff polynomial for a compact set* $\mathcal{E}$. *That is to say the unique polynomial for which*

$$M_n \equiv \inf_{p_n(z) \in P_n} \sup_{z \in \mathcal{E}} |p_n(z)|,$$

where P_n *is the set of all* $n\underline{th}$ *order monic polynomials. Then the capacity of* $\mathcal{E}$ *is,*

$$\operatorname{cap}(\mathcal{E}) \equiv \lim_{n\to\infty} [M_n(\mathcal{E})]^{1/n}.$$

The study of convergence in measure was pioneered by Nuttall [4] in 1970. He proved

NUTTALL'S THEOREM. *Let $f(z)$ be analytic at the origin and also in the circle $|z| \leq R$ except for m poles, counting multiplicity. (He makes no further assumptions on f outside the closed circle of radius R.) Consider a sequence $[L_k/M_k]$ of Padé approximants to $f(z)$ with $M_k \geq m$, and $L_k/M_k \to \infty$ as $k \to \infty$ $(M_k \neq 0)$. Let ϵ and δ be arbitrarily small positive, given numbers. Then k_0 exists such that*

$$|f(z) - [L_k/M_k]| < \epsilon,$$

for all $k > k_0$ and for all $|z| < R$ except for $z \in \mathcal{E}$, where $\mathcal{E}$ is a set of points in the z-plane of measure less the δ.

POMERENKE'S THEOREM, [5]. *Let $f(z)$ be regular in $|z| \leq R$ for all $0 < R < \infty$ except for a finite number (counting multiplicity) of isolated poles $\nu(R)$ and essential singularities $\mu(R)$. Then, given any ϵ, $\delta > 0$ and $1 \geq \lambda > 0$, there exists an M_0 such that for all $M > M_0$, $\lambda^{-1}M > L > \lambda M$,*

$$|f(z) - [L/M]| < \epsilon,$$

for all z in a given, closed, bounded region $\mathcal{R}$ of the complex plane except for a set of points $\mathcal{E}^{[L/M]}$ of measure less than δ.

The next big step in this approach was taken by Stahl (1987-97) [6]. First of all he answered definitively the question of where do the cuts go, when a multivalued function is approximated. The answer is given by

STAHL'S MINIMAL SET THEOREM. *Let $f(z)$ be a given functional element, analytic in the neighborhood of zero. There exists a unique compact set $\kappa_0 \in \mathcal{C}(w)$, $w = 1/z$ such that*

(i) $\mathcal{D}_0 = \hat{\mathcal{C}} \backslash \kappa_0$ is the domain in which $f(z)$ has a single-valued analytic continuation,

(ii) with respect to the w-plane, $\mathrm{cap}(\kappa_0) = \inf_\kappa \mathrm{cap}(\kappa)$, where the infimum extends over all compact sets $\kappa \subseteq \mathcal{C}(w)$ satisfying (i), and

(iii) $\kappa_0 \subseteq \kappa$ for all compact sets $\kappa \subseteq \mathcal{C}(w)$ satisfying (i) and (ii).

The notations $\mathcal{C}(w)$ and $\hat{\mathcal{C}}(w)$ refer to the complex w-plane and the complex w-plane including the point at infinite respectively.

The set κ_0 is called the *minimal set* (for single-valued analytic continuation of $f(z)$) and the domain $\mathcal{D}_0$ is called the *extremal domain.* This result for the unique minimal set provides the precise description of the set that Nuttall [7] conjectured should exist and should be the cut-set for the given function.

In addition Stahl has proven convergence in capacity. To state his theorem, we define,

$$F_f(z) = \exp\{-g_{\mathcal{D}_0}(z,0)\}, \quad z \in \mathcal{D}_0,$$

where $g_{\mathcal{D}_0}(z,0)$ is the Green's function of the domain $\mathcal{D}_0$ with a logarithmic singularity at $z = 0$.

Stahl's Convergence in Capacity Theorem. *Let $f(z)$ be given by a functional element at $z = 0$ and let the set $\mathcal{E} \subset \hat{\mathcal{C}}(w)$, $(w = 1/z)$ of all the singularities of $f(z)$ be of capacity $\operatorname{cap}(\mathcal{E}) = 0$ with respect to the w-plane. Then any close-to-diagonal sequence of Padé approximants $\{[L/M](z)\}$ to the function $f(z)$ converges in capacity to $f(z)$ in the extremal domain $\mathcal{D}_0$. More precisely: For any compact set $\mathcal{V} \subseteq \mathcal{D}_0$, and $\epsilon > 0$, we have (with respect to the w-plane)*

$$\lim_{L+M\to\infty} \operatorname{cap}\{z \in \mathcal{V} \,|\, |f(z) - [L/M](z)| > [F(z) + \epsilon]^{L+M}\} = 0,$$

and

$$\lim_{L+M\to\infty} \operatorname{cap}\{z \in \mathcal{V} \,|\, |f(z) - [L/M](z)| < [F(z) - \epsilon]^{L+M}\} = 0$$

An infrequently mentioned theorem shows that if a natural boundary is not too thick, the Padé approximants can analytically (in the sense of Borel and Carleman) continue through it and converge beyond the boundary [8]:

Gammel and Nuttall's theorem. *Let $f(z)$ be quasi-analytic, i.e.,*

$$f(z) = \sum_{n=1}^{\infty} \frac{A_n}{1 - z\alpha_n}, \qquad |A_n| < Ce^{-n^{1+\epsilon}}, \quad \epsilon > 0$$

and let $|\alpha_n| = 1$. Then the sequence of $[N+J/N]$ Padé approximants to $f(z)$ converges in measure to f, $N \to \infty$ in any closed bounded region of the complex plane. J is any N independent integer. (By sequence is meant all those which exist. There are always an infinite number.)

In the first approach, there are a number of special cases which can be analyzed analytically. The location of cuts and rates of convergence are known. There are in these cases no defects. In the second approach a rather wide class of functions is analyzed. There are explicit methods to determine the location of the cuts, and the rate of convergence. The class is not universal however, as it does not include functions with natural boundaries nor asymptotic series. In the results for this approach defects are the norm. From a practical point of view, one would like to be assured that there are no defects, so one could take a given approximant and evaluate it anywhere in the cut-plane and get an accurate result.

The third approach is that of pointwise convergence of a subsequence. Just as in sorting a barrel of apples, the idea is simply to throw away the bad ones. What remains to show is basically that there will be an infinite number of good approximants left. So far, progress, though considerable is not as great as in the second approach. To get some orientation to the problem, it is instructive first to consider some counter-examples.

PERRON'S EXAMPLE [9]. Let

$$f(z)=\sum_{n=0}^{\infty} c_n z^n.$$

Select a sequence of points z_n, dense in the complex plane. The values $z_n = 0$ or ∞ are not used for any n. On the basis of this sequence, define

$$\left.\begin{array}{c} c_{3n}=z_n/(3n+2)! \\ c_{3n+1}=c_{3n+2}=1/(3n+2)! \end{array}\right\} \text{ if } |z_n|\le 1,$$

$$\left.\begin{array}{c} c_{3n}=c_{3n+1}=1/(3n+2)! \\ c_{3n+2}=z_n^{-1}/(3n+2)! \end{array}\right\} \text{ if } |z_n| > 1.$$

By comparison with the exponential series, as $|c_j|\le 1/j!$ for all j, the function $f(z)$ is an entire function, (has an infinite radius of convergence). But as either $c_{3n}/c_{3n+1}=z_n$ or $c_{3n+1}/c_{3n+2}=z_n$, as the $|z_n|$ is greater or less than unity, and as

$$[L/1]=\sum_{j=0}^{L-1} c_j z^j + \frac{c_L z^L}{1-\frac{c_{L+1}}{c_L}z},$$

we see that either the $[3n/1]$ or $[3n{+}1/1]$ has a pole at $z=z_n$. Therefore the entire sequence has poles (defects) dense in the complex plane; however, the $[3n+2/1]$ sequence does converge.

GAMMEL-WALLIN EXAMPLE [10]. Let $n_1=1$, $n_{k+1}=2n_k+1$ $(=2^k-1)$. Let the series coefficients c_n now be chosen as $c_n=\alpha_k z_k^{-n}$ when $n_k\le n<n_{k+1}$. Again choose a sequence of $z_k\ne 0,\infty$ dense in the complex plane and pick

$$\alpha_k=\frac{\min(|z_k|^{n_k},|z_k|^{2n_k})}{(2n_k)!}.$$

Again we have the property that $|c_n| \leq 1/n!$, and so again $f(z)$ is an entire function. We may write, by partially summing up the series,

$$f(z) = 1 + \sum_{k=1}^{\infty} \alpha_k \frac{\left(\frac{z}{z_k}\right)^{n_k} - \left(\frac{z}{z_k}\right)^{2n_k+1}}{1 - \left(\frac{z}{z_k}\right)}.$$

Inspection shows that

$$[n_k/1] = [n_k + 1/1] = \cdots = [2n_k - 1/1],$$

and therefore by Padé's block theorem, there is an $n \times n$ block starting at the $[n_k/1]$. Hence $[n_k/1] = [n_k/n_k]$ has a pole (defect) at z_k. Thus the whole $[n/n]$ sequence (and in particular the $[2^k - 1/2^k - 1]$ subsequence) diverges on an everywhere dense subset of the complex plane! (Note that there does exist another diagonal subsequence which converges everywhere.)

Buslaev, Gončar, and Suetin's example [11]. Consider the function,

$$f(z) = \frac{1 + \sqrt[3]{2}z}{1 - z^3} = 1 + \sqrt[3]{2}z + z^3 + \sqrt[3]{2}z^4 + \cdots.$$

This function is regular in $|z| < 1$ and also on the unit circle, except at the three cube roots of unity. Never-the-less, every approximant of the type $[L/2]$ has a pole (defect) in $|z| < 1$. Note that the $[1/3]$ is exact.

Lubinsky and Saff's example [12]. Consider the partial theta function $h_q(z)$ with $q = e^{i\theta}$ where $\theta/(2\pi)$ is irrational.

$$h_q(z) = \sum_{i=1}^{\infty} q^{i(i-1)/2} z^i$$

This function is clearly regular for $|z| < 1$ for our choice of q Furthermore, there is a natural boundary at $|z| = 1$. They show that the Padé approximants $[L/M]$ with M fixed, converge in a disk $|z| < \Delta_{M,q} < 1$. However, for $M \geq 2$ every approximant has a pole (defect) with $|z| < 1$ and the boundary of $\Delta_{M,q}$ contains limit points of poles of the row sequence. Note that they also show a proper subsequence of the $[M/M]$ converges locally uniformly in all of $|z| < 1$.

If $\theta/(2\pi) = p/q$ is rational, then since $[i(i-1)/2] \operatorname{mod}_q$ is periodic with finite period $\nu(q) \leq 2q$, the coefficients of $h_q(z)$ are also periodic. Thus

$$h_q(z) = \frac{P_{\nu(q)}(z)}{1 - z^{\nu(q)}},$$

and so the $[\nu - 1/\nu]$ is exact. In light of this result, it seems likely that in the irrational case, the natural boundary is a permeable one and that the Padé approximants converge in measure beyond it.

These counter examples demonstrate clearly that convergence of the whole sequences is not to be expected in general, so for the third approach we focus on subsequence convergence. Here follow some of the positive results that have been obtained. They have restrictions of various types. Some are restrictions on the analytic properties of the functions, some on the behavior of the series coefficients, some on the behavior of the approximants, and some are of mixed type. First we report on theorems on the convergence of general types of sequences of Padé approximants. The most primitive such theorem is due to Baker [13].

BOUNDEDNESS THEOREM. *Let $P_k(z)$ be any sequence of $[L/M]$ Padé approximants to a formal power series where $L+M \to \infty$ with k. If $|P_k(z)|$ is uniformly bounded in any closed, simply-connected domain D_1 containing the origin as an interior point and $|P_k(z)|^{-1}$ is likewise in D_2, then the $P_k(z)$ converge to a meromorphic function in the interior of $D_1 \cup D_2$.*

This theorem can be stated in a considerably improved manner in terms of the Riemann sphere. Consider a sphere with the unit circle of the complex plane as its equator. Run a line from its north pole to any point on the complex plane. The single intercept of this line with the sphere is Riemann's spherical representation of the complex plane. The north pole itself is the point at infinity. Let the distance between any two points be the length of the cord between them. This distance is always less than 2 clearly. We need the derivative, in terms of this cordal distant, of $f(z)$. It is,

$$\frac{Df}{Dz} = \frac{2|f'(z)|}{1+|f(z)|^2}.$$

Combining some results, we have [14],

BAKER AND GRAVES-MORRIS'S THEOREM. *If $P_k(z)$ is any sequence of Padé approximants for which $L+M \to \infty$ with k, then it is necessary and sufficient that $D\,P_k(z)/Dz$ be uniformly bounded in $\mathcal{R}$, a closed region containing the origin as an interior point, in order that $\{P_k(z)\}$ converge in $\mathcal{R}$ to a limit on the sphere. That limit is a meromorphic function.*

What if there are no spare poles in the unit disk?

BEARDON'S DISK THEOREM [15]. *Let $f(z)$ be analytic in $|z| < R$ except for m nonzero poles there, and let $\mathcal{E}$ be any closed set of $|z| < R$ on which $f(z)$ is analytic. For each $\delta > 0$ there exist a k ($k \geq 1$ and depends only on f, $\mathcal{E}$, and δ) such that if $[L/M]$ is any sequence of*

Padé approximants to $f(z)$ whose poles are at least δ from $\mathcal{E}$ and for which $L \geq kM \geq m$, then $[L/M]$ converges uniformly to $f(z)$ on $\mathcal{E}$ as $L + M \to \infty$.

BAKER'S DISK THEOREM [16]. *Let $f(z)$ be given, regular and non-zero at $z = 0$, and meromorphic in $|z| < S$ with l zeros and m poles. Let $\{P_k(z)\}$ be any sequence of Padé approximants such that $L + M \to \infty$ with k, and such that there are exactly l zeros and m poles of $\{P_k(z)\}$ in $|z| < S$, then $\{P_k(z)\}$ converges to $f(z)$ in $\{z \,|\, |z| < S \setminus \{$poles of $f(z)\}\}$.*

The previous theorem is a corollary of the following more general theorem [16].

BAKER'S THEOREM. *Let $f(z)$ be a given function with l zeros and m poles (counting multiplicity), meromorphic in an open, simply-connected region $\mathcal{R}$ of the complex plane containing the origin. Let $\{P_k(z)\}$ be a sequence of $[L/M]$ Padé approximants to $f(z)$, $L + M \to \infty$ with k, such that the sum of the number of poles and zeros $n_k(d)$ of $P_k(z)$ in $\mathcal{R}$ and more distant than any $d > 0$ from the boundary of $\mathcal{R}$ satisfies,*

$$\lim_{k\to\infty} \left[\frac{n_k(d)}{L_k + M_k} \right] = 0.$$

Further, let $\mathcal{T}$ be an arbitrary closed set interior to $\mathcal{R}$. (Select d so that none of $\mathcal{T}$ is closer to the boundary than d.) Let there be exactly l_1 zeros and m_1 poles (counting multiplicity) of $f(z)$ in the interior of $\mathcal{T}$ and an equal number of poles and zeros of $\{P_k(z)\}$ in $\mathcal{T}$ and no other limit point of poles or zeros in $\mathcal{T}$. Then the sequence $\{P_k(z)\}$ converges in compact subsets of $\{\mathcal{T} \setminus \{$poles of $f(z)\}\}$ to $f(z)$. The conditions on $\{P_k(z)\}$ are also necessary, provided $\mathcal{T}$ contains the origin as an interior point and contains no limit point of external poles or zeros.

Another theorem, proven earlier can also be treated as a corollary of the previous theorem. It is [17]:

WALSH'S THEOREM. *Let Δ be a Jordan region of the extended plane containing the origin, whose boundary is denoted by Γ. Let $w = \phi(z)$, map Δ conformally and one-to-one onto $|w| < 1$ and let Γ_m denote generically the locus $|\phi(z)| = m$, $0 < m < 1$, in Δ. Let $f(z)$ be analytic at the origin and on Γ, and suppose that the Padé approximants $\{P_k(z)\}$, where $L + M \to \infty$ with k, are bounded on Γ:*

$$|f(z) - P_k(z)| \leq \mathcal{M}, \quad z \quad \text{on} \quad \Gamma$$

Suppose $P_k(z)$ has precisely N_k poles in Δ, and $N_k/(L+M) \to 0$. Then we have

$$\limsup_{k\to\infty} \{\max\left[D(f(z), P_k(z))\right], \; z \text{ on } T\}^{1/(L+M)}$$

$$\leq \max |\phi(z)|, \ z \text{ on } T,$$

where $D(x,y)$ is the cordal distance on the sphere, T is an arbitrary, closed set in Δ containing only one pole of $P_k(z)$ for each pole of $f(z)$ in the interior of T, and no other limit point of poles.

There is another group of theorems where the assumptions do not involve pole location, but instead have more restrictive assumptions on the function class considered.

BAKER AND GAMMEL'S THEOREM[18, 19]. *Let $f(z)$ be meromorphic in the whole complex plane. Then there exists an infinite subsequence of $[N+j/N]$ Padé approximants which converge to $f(z)$ at any point of the complex plane not a pole of $f(z)$, provided, that if a_n are the locations of the poles of $f(z)$ ($|a_n| \leq |a_{n+1}|$) that j goes to infinity sufficiently rapidly so that $\max_{1\leq n\leq N}\{b_n^{-1}|a_n/a_{N+1}|^j\} \to 0$ as $N \to \infty$. We further assume that if the b_n are the residues that $\sum |b_n/a_n|$ converges.*

We define,

$$\vec{\phi}_i = \mathbf{A}^{i-1}\vec{g}, \quad \vec{\phi}'_i = (\mathbf{A}^\dagger)^{i-1}\vec{h}$$

where $\mathbf{A}^\dagger$ is the Hermitian conjugate of the operator $\mathbf{A}$. We also define the $N \times N$ matrix,

$$R_{i,j} = (\vec{\phi}'_i, \vec{\phi}_j) = (\vec{h}, \mathbf{A}^{i+j-2}\vec{g}) \equiv \omega_{i+j-2}$$

If $\det |R_{i,j}| \neq 0$, we can define the projection operator

$$\mathbf{P}_N = \sum_{i,j=1}^{N} \vec{\phi}_i (R^{-1}_{i,j}) \vec{\phi}'_j.$$

$\mathbf{P}_N$ is projection operator on the space $\mathcal{S}_N$ spanned by the ϕ_i. As this projection operator may however be oblique ($\| \mathbf{P}_N \| > 1$), it is convenient to define in addition the corresponding orthogonal projection operator, $\mathcal{P}_N$. Correspondingly we define the orthogonal projection operator $\mathcal{P}'_N$ which projects onto the space $\mathcal{S}'_N$ which is spanned by the ϕ'_j. It is now easy to show that, in terms of the solution of

$$\vec{f}_N = \vec{g} + \lambda \mathbf{P}_N \mathbf{A} \mathbf{P}_N \vec{f}_N$$

the $[N-1/N]$ Padé approximant to $(\vec{h}, \vec{f})$ where

$$\vec{f} = \vec{g} + \lambda \mathbf{A}\vec{g} + \lambda^2 \mathbf{A}^2 \vec{g} + \cdots,$$

is just $(\vec{h}, \vec{f}_N)$. If $\vec{g} = \vec{h}$ and $\mathbf{A}$ is Hermitian, then from Fredholm theory we know that the solution $(\vec{h}, \vec{f})$ is a meromorphic function in the complex λ-plane.

Baker's functional equation theorem [20]. *Let $f(z)$ be defined as above in terms of $\mathbf{A}$, $\vec{g}$, then (i) if $\mathbf{A}$ is of trace class, [i.e., $\mathbf{A}^\dagger\mathbf{A}\psi_i = \alpha_i^2\psi_i$, $\parallel \mathbf{A} \parallel_1 = \sum_{i=1}^\infty \alpha_i$] then the numerator and the denominator of the $[M/M]$ Padé approximants converge separately to entire functions throughout the complex λ-plane and their ratio converges strongly to the $f(\lambda)$. (ii) If $\mathbf{A}$ is a compact operator and*

$$\liminf_{N\to\infty} \parallel \mathcal{P}_N \mathbf{A} \mathbf{P}_N \mathbf{A} \vec{e}_N \parallel = 0$$

where $\vec{e}_N$ is the unit vector in $\mathcal{S}_N$ but not in $\mathcal{S}_{N-1}$, then either $\mathcal{P}'_N \vec{f}_N$ converges strongly to the solution of the functional equation, with the exception of its singular points and at most one other point, or there exist two infinite subsequences for one or the other or which $\mathcal{P}'_N \vec{f}_N$ converges strongly (except at singular points of the functional equation) for all finite λ. Since $\vec{h} \in \mathcal{S}'_N$, these results imply the corresponding convergence of the $[N-1/N]$ Padé approximants.

Next we report a line of work begun perhaps by Dumas [21].

Dumas theorem. *Let*

$$f(z) = \sqrt{(z-d_1)(z-d_2)(z-d_3)(z-d_4)}$$

where the d_i are distinct. Then, there exists a subsequence of the $[M/M]$ Padé approximants which converge point by point in the domain give by Stahl's minimal set theorem.

Of course Dumas used other language as he did not have Stahl's result at that time, but this is a correct statement of some of his results. Nuttall [22] has extended these results. In Dumas theorem, the branch cuts consist of two arc each of which joins a pair of branch points. There can be at most one spurious pole. Nuttall has generalized this theorem to:

Nuttall's branch-point theorem. *Let*

$$f(z) = \int_S dt \left\{ \prod_{i=1}^{2l} (t-d_i) \right\}^{-\frac{1}{2}} \mu(t)(t-z)^{-1}$$

where the d_i are distinct and a particular choice of $\mu(t)$. Then outside a cut-set which connects (in pairs) the branch-points and has the minimum capacity in the $1/z$-plane, there is a subsequence of the $[M/M]$ Padé approximants which converges point by point to $f(z)$.

In this case, the description of the cut-set agrees with Stahl's results. Also here there can be at most $(l-1)$ spurious poles. The contour of integration S is appropriately chosen. Stahl [23] has reported proving the same results for a subclass of the hyper-elliptic functions.

Lubinsky [24] has proven a number of theorems which depend on smoothness properties of the coefficients of the series expansion. I have consolidated them into a single theorem.

LUBINSKY'S THEOREM. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be entire. (i) If infinitely many $a_j \neq 0$, and*

$$\limsup_{L\to\infty} |a_L|^{1/L^2} = \rho < \frac{1}{3},$$

then there exists an infinite sequence S of positive integers such that

$$\lim_{\substack{L\to\infty \\ L\in S}} [L/M_L](z) = f(z)$$

uniformly in compact subsets of the finite complex plane for all non-negative integer sequences $\{M_L\}_{L=0}^{\infty}$. (ii) If

$$|a_{j-1}a_{j+1}/a_j^2| \leq \rho^2, \quad j = 1, 2, 3, \ldots,$$
$$2\sum_{j=1}^{\infty} \rho^{j^2} = 1, \quad \rho \approx 0.4559\ldots,$$

then for any sequence of non-negative integers $\{M_L\}_{L=1}^{\infty}$, the $[L/M_L](z)$ converge uniformly in compact subsets of the finite complex plane to $f(z)$ (iii)If $a_j \neq 0$, j large enough, and for some complex number q, $|q| < 1$

$$\lim_{j\to\infty} a_{j-1}a_{j+1}/a_j^2 = q,$$

then for any sequence of Padé approximants $[L_k/M_k]$ where $L_k \to \infty$ as $k \to \infty$ converges locally uniformly in the finite complex plane. In each case the diagonal sequence is included.

Finally there is the Baker-Gammel-Wills [2] (or Padé) conjecture. There are by now many inequivalent variants of this conjecture. I will stick with the original version. An important point is the implied existence of a singularity at $z = +1$.

PADÉ CONJECTURE. *If $P(z)$ is a power series representing a function which is regular for $|z| \leq 1$, except for m poles within this circle and except for $z = +1$, at which point the function is assumed continuous when only points $|z| \leq 1$ are considered, then at least a subsequence of the $[N/N]$ Padé approximants converge uniformly to the function (as N tends to infinity) in the domain formed by removing the interiors of small circles with centers at these poles.*

I would remark that the reason for the continuity condition at the boundary singularity is illustrated by the following two examples. Let

$$a(x) = (1 - e^{-x})/x, \qquad b(x) = (1 - e^{-x}).$$

Then,

$$[N/N]_a(\infty) = \frac{(-1)^N}{N+1} \to 0,$$
$$[N/N]_b(\infty) = 1 - (-1)^N \; ? \to 1$$

In case $a(z)$ the continuity condition is satisfied and the limit exists. Here a majority of directions have a single limit. In case $b(z)$ the continuity condition fails and the whole sequence oscillates indefinitely about the limit found as $z \to \infty$ along the real axis.

Now we turn to results on more or less horizontal (or vertical by duality) sequences. The first and best known is

DE MONTESSUS'S THEOREM [25]. *Let $f(z)$ be a function which is meromorphic in the disk with $|z| \leq R$, with precisely m poles at distinct points, $z_1, z_2, \ldots, z_m$ with*

$$0 < |z_1| \leq |z_2| \leq \ldots \leq |z_m| < R.$$

Let the pole at z_k have multiplicity μ_k, and let the total multiplicity be $\sum_{k=1}^{m} \mu_k = M$ precisely. Then

$$f(z) = \lim_{L \to \infty} [L/M],$$

uniformly on any compact subset of $\mathcal{D}_m = \{z \,|\, |z| < R, z \neq z_k, k = 1, \ldots, m\}$.

BEARDON'S ROW THEOREM [26]. *Let $f(z)$ be analytic in $|z| \leq R$. Then an infinite subsequence of the $[L/1]$ Padé approximants converges to $f(z)$ uniformly in $|z| \leq R$.*

BUSLAEV, GONČAR, AND SUETIN'S THEOREM [11]. *Let $f(z)$ be holomorphic at $z = 0$ and be given by its power series expansion. Let R_M be its radius of meromorphy with no more than M poles. Then, there exists at least a subsequence of $[L/M]$ which converges uniformly as $L \to \infty$ on compact subsets of the disk $|z| < c_M R_M$ which do not contain poles of f. c_M is a positive constant which depends only on M. If $R_M = \infty$ then we get convergence in the whole complex plane excluding the poles of $f(z)$.*

The counter-example of Lubinsky and Saff [12] shows that $c_2 \leq 0.58\ldots,$

$c_{17} \leq 0.24\ldots$ and there are M's where the $c_M \leq 3 - 2\sqrt{2}$. We will see below that the behavior on the circle of M-meromorphy makes a big difference in what can be proven!

LUBINSKY'S ROW THEOREM [27]. *Let* $f(z) = \sum_{m=0}^{\infty} a_m z^m$ *satisfy the properties*

$$\liminf_{m\to 0} |a_m/a_{m+1}| = R > 0, \quad q_m = a_{m-1}a_{m+1}/a_m^2,$$
$$\lim_{m\to\infty} q_m = q, \quad q \neq 0, \quad q^j \neq 1 \;\forall\, 0 < j < n,$$

or, if q *is a root of unity, there exists the expansion,*

$$q_m = q \sum_{k=1}^{N} c_k m^{-k} + o\left(m^{-N}\right) \quad m \to \infty.$$

then

$$\lim_{m\to\infty} [m/n] = f(z), \quad |z| < \sigma R,$$

where σ *is the smallest root (necessarily positive) of the Rogers-Szegő polynomial*

$$B_n(-u) = \sum_{j=0}^{n} \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n+1-j})}{(1-q)(1-q^2)\cdots(1-q^j)} u^j.$$

Note that if $q = 1$, *then* $\sigma = 1$ *for all* n.

So far as I can tell the first work on Padé approximants to "smooth" series is due to Wilson [28]. Here is an extended version [29] of one of his theorems.

WILSON'S THEOREM. *If* $f(z)$ *inside* $|z| < 1$, *except for* n *interior poles* r_i *with corresponding multiplicities* m_i, $\sum_{i=1}^{n} m_i = M$ *then if the coefficients of*

$$B(z) = \sum_{j=0}^{\infty} b_j z^j = f(z) \prod_{i=1}^{n} (1 - z/r_i)^{m_i}$$

satisfy the conditions for $n = L - M - \mu + 1$ *to* $L + M + \mu$

$$b_n = \Gamma^{-n}\beta(L)\left[\sum_{j=0}^{2\mu-2} \alpha_j L\left(\tfrac{n}{L}-1\right)^j + \gamma(n,L)\left(\tfrac{M+\mu}{L}\right)^{2\mu-1}\right],$$

where $|\Gamma| = R$ *and* $\alpha_j(L)$ *and* $\gamma(n,L)$ *are uniformly bounded as* $L \to \infty$, *and*

$$\lim_{L\to\infty} \det \begin{vmatrix} (2\mu-2)!\alpha_{2\mu-2}(L) & \cdots & (\mu-1)!\alpha_{\mu-1}(L) \\ \vdots & \ddots & \vdots \\ (\mu-1)!\alpha_{\mu-1}(L) & \cdots & \alpha_0(L) \end{vmatrix}$$

exists and is not zero, then in the limit as $L \to \infty$, $[L/M+\mu]$ *converges uniformly to* $f(z)$ *on compact subsets of* $|z| < R$ *excluding poles of* $f(z)$.

Wallin's theorem [30]. *If $f(z)$ satisfies the conditions of Wilson's theorem for all μ or the conditions of Lubinsky's row theorem, then there exists an infinite sequence of $[L/M]$ Padé approximants which converge uniformly on compact subsets of $|z| < R$ excluding poles of $f(z)$. This sequence has the property that $L_i \to \infty$ and $M_i \to \infty$ as $i \to \infty$.*

Looking at functions of the class of the Padé conjecture, I have found them to have certain valuable smoothness properties by using some century old results. In this class of functions, there is a single non-polar singularity on the boundary of the circle of M-meromorphy. The following *new* theorem will be stated for no poles in the disk, but it can be easily extended by standard methods of proof to a specific, finite number of poles in the disk.

Baker's row theorem. *If $P(z)$ is a power series representing a function which is regular for $|z| \leq 1$, except for $z = +1$, at which point the function is assumed continuous when only points $|z| \leq 1$ are considered, then*

$$\lim_{L\to\infty} [L/m] = f(z), \quad |z| \leq 1 \setminus z = +1.$$

Proof: Since $f(z)$ is regular on $|z| = 1 \setminus z = +1$ it is also finite at each such point. If $|f(1)| = \infty$, then the oscillation in any finite interval including $z = +1$ is infinite. However as $f(z)$ is assumed continuous at $z = +1$, it must also be bounded there, and consequently bounded in $|z| \leq 1$ Thus the singularity at $z = +1$ is non-polar. It must be either a branch-point or an essential singularity. By the Heine-Borel covering theorem, we can cover the line $z = e^{i\theta}$, $\delta \leq \theta \leq 2\pi - \delta$ of regular points with a finite number of interiors of circles of regularity. Therefore we can enclose this line in a circle of regular points. (It will pass just inside the point $z = +1$.) By Cauchy's theorem, we can define $g(n)$ such that $g(n) = a_n$, $n = 0, 1, 2, \ldots$ by the formula,

$$g(n) = \frac{1}{2\pi i} \oint \frac{f(z)\, dz}{z^{n+1}}.$$

where the contour of integration is the aforementioned circle. From this definition, it is clear that $g(n)$ is regular for $\Re(n) > 0$. It is now convenient to compute a bound for the derivatives of $g(n)$.

$$\begin{aligned} g^{(j)}(n) &= \frac{1}{2\pi i n^j} \oint \frac{(n \ln z)^j f(z)\, dz}{z^{n+1}} \\ &= \frac{1}{2\pi i n^{j+1}} \oint (n \ln z)^j \exp(-n \ln z) f(z)\, d(n \ln z). \end{aligned}$$

we can treat the essential features of a bound by using the contour $z = [1 + \frac{\delta}{2}(1 - \cos\theta)]e^{i\theta}$ where $-\pi \leq \theta \leq \pi$. The variable $x = n \ln z$ runs from $+\infty - i\sqrt{\infty}$ through 0 to $+\infty + i\sqrt{\infty}$. If we replace $f(z)$ by its bound, K, on the contour and recognize what is left in terms of x as the standard gamma function integral, then we get

$$\left| g^{(j)}(n) \right| \leq \frac{j!K}{\pi n^{j+1}}, \quad n \to \infty.$$

From this result we may compute by expanding $g(n)$ about $n = m$ through the use of Taylor's theorem with remainder, that

$$\begin{aligned} \frac{a_{m-1}a_{m+1}}{a_m^2} &= \left[1 + \sum_{j=1}^{N} (-1)^j b_j m^{-j} + o\left(m^{-N}\right) \right] \\ &\quad \times \left[1 + \sum_{j=1}^{N} b_j m^{-j} + o\left(m^{-N}\right) \right]. \end{aligned}$$

Note that the "smoothness" of the coefficients is intimately related to the analyticity properties. Thus, in the notation of Lubinsky's row theorem, we see that $q = 1$ so that $\sigma = 1$. In addition, we have just seen that the necessary expansion property holds as the radius of regularity about $n = m$ is at least m. Thus the conclusions of this theorem hold in $|z| < 1$ by Lubinsky's row theorem. Since in the course of Lubinsky's proof [27], he also shows that the poles of the approximant converge to unity, the numerator must converge to $(1 - z)^m f(z)$ which is bounded in $|z| \leq 1$. Take note that by our inequalities on the series coefficients, by a theorem of Fatou [31], the series converges at all points on the unit circle except $z = 1$. More specifically, the numerator is

$$P_L(z) = f(z) \left[(1 - z)^m + \sum_{k=1}^{m} \epsilon_{L,k}(1 - z)^{m-k} \right] \Bigg|^L,$$

where the ϵ's tend to zero as $L \to \infty$ and the notation $|^L$ means truncate to L terms. The series coefficients of the untruncated numerator series obey the inequality $|lp_l| < C$ as shown above for the coefficients of $f(z)$ and from the assumptions of the theorem the function defined by the untruncated numerator series is bounded on the unit disk, it follows by a theorem of Dienes [32] that the numerator is also bounded on the unit disk. Thus, by Vitali's theorem [33], the approximants converge uniformly on compact subsets of the unit disk, except the point $z = +1$. If converges fails at this exceptional point, the value can be obtained by continuity from adjacent converged points. ∎

Wallin's theorem applies to these results as well.

The method of proof employed in the proof of this theorem, could equally well have been applied to extend Wilson's theorem, but it seems that the above theorem includes what could have been derived as an extension of Wilson's theorem in a more simple way. Never-the-less the arguments and techniques used to include the case of a finite number of poles in the interior of the disk for Wilson's theorem, can be applied to this case in a fairly straight forward manner.

The significant difference between this current theorem and the counter examples of Buslaev, Gončar & Suetin, and Lubinsky & Saff, is that there is exactly one singular point on the circle of meromorphy; This singularity provides an unambiguous place to which the extra poles can converge.

Also we know from the theorems of Leau [34] and Faber [35], and of Wigert [36] and Faber [35] that a uniform function with an isolated singular point corresponds to the coefficients as a function of the suffix being regular in a half-plane and of limited growth. The converse is true as well. These are further examples of how the "smoothness" is related to analyticity.

References

1. Baker, G. A., Jr.; and Graves-Morris, P. R., *Padé Approximants, Second Edition, Encyclopedia of Mathematics and its Applications* **59**, G.-C. Rota, ed., Cambridge Univ. Press, New York, 1996.
2. Baker, G. A., Jr.; Gammel, J. L.; and Wills, J. G.: 1961, An Investigation of the Applicability of the Padé Approximant Method, *J. Math. Anal. Appl.* **2**, pp. 405-418.
3. Cartan, H.: 1928, Sur les systèms de fonctions holomorphes à variétés linéaires et leurs applications, *Ann. Sci. École Norm. Sup.* (3) **45**, pp. 255-346.
4. Nuttall, J.: 1970, The Convergence of Padé approximants of meromorphic functions, *J. Math. Anal. Appl.* **31**, pp. 147-153.
5. Pomerenke, C.: 1973, Padé approximants and convergence in capacity, *J. Math. Anal. Appl.* **41**, pp. 775-780.
6. Stahl, H.: 1987, Three different approaches to a proof of convergence for Padé approximants, in *Rational Approximation and its Applications in Mathematics and Physics*, eds. J. Gilewicz, M. Pindor and W. Sienaszko, *Lecture Notes in Mathematics, no. 1237* Springer, Berlin, pp. 79-124; — : 1997, The convergence of Padé approximants to functions with branch points, *J. Approx. Theory* **91**, pp. 139-204.
7. Nuttall, J.: 1984, Asymptotics of diagonal Hermite-Padé polynomials, *J. Approx. Theory* **42**, pp. 299-386.
8. Gammel, J. L.; and Nuttall, J.: 1973, Convergence of Padé Approximants to Quasianalytic Functions Beyond Natural Boundaries, *J. Math. Anal. Appl.* **43**, pp. 694-696.

9. Perron, O.: 1957, *Die Lehre von den Kettenbrüchen* Vol. 2., Teubner, Stuttgart, Chap. 4.
10. Gammel, J. L.; 1970, private communication; Wallin, H., On the convergence theory of Padé approximants in *Linear Operators and Approximation, (Proc. Conf. at Oberwolfach)*, P. L. Butzer, J.-P. Kahana, and B. Sz.-Nagy, eds., Birkhäuser, Basel, 1972,
11. Buslaev, V. I.; Gončar, A. A.; and Suetin, S. P.: 1983. On convergence of subsequences of the mth row of the Padé table, *Math. USSR Sbornik* **48**. pp. 535-540, [*Mat. Sbornik* **120**(162)].
12. Lubinsky, D. S.; and Saff, E. B.: 1987, Convergence of Padé approximants of partial theta functions and Rogers-Szegő polynomials, *Const. Approx.* **3**, pp. 331-336.
13. Baker, G. A., Jr.: 1965, The Theory and Application of the Padé Method, *Adv. Theor. Phys.* **1**, pp. 1-58, Academic Press, New York.
14. Baker, G. A., Jr. and Graves-Morris, P. R.: 1982, The Convergence of Sequences of Padé Approximants, *J. Math. Anal. Appl.* **87**, pp. 382-394.
15. Beardon, A. F.: 1968, The convergence of Padé approximants, *J. Math. Anal. Appl.* **21**, pp. 344-346.
16. Baker, G. A., Jr.: 1976, A theorem on the convergence of Padé approximants, *Studies in Applied Math.* **55**, pp. 107-117.
17. Walsh, J. L.: 1967, On the Convergence of sequences of rational functions, *SIAM, J. Numer. Anal.* **4**, pp. 211-221.
18. Baker, G. A., Jr.; and Gammel, J. L.: 1961, The Padé approximant, *J. Math. Anal. Appl.* **2**, pp. 21-30.
19. Baker, G. A., Jr.: 1970, The Padé Approximant Method and Some Related Generalization in *The Padé Approximant in Theoretical Physics*, G. A. Baker, Jr. and J. L. Gammel, eds., pp. 1-39, Academic Press, New York.
20. Baker, G. A., Jr.: 1975, Convergence of Padé approximants using the solution of linear functional equations, *J. Math. Phys.* **16**, pp. 813-822.
21. Dumas, S.: 1908, Sur le développement des fonctions elliptique en fractions continues, *thesis* Faculté de Philosophie, University of Zürich.
22. Nuttall, J.: 1973, Variational principles and Padé approximants, in *Rational Approximation*, E. B. Saff and R. S. Varga, eds., Academic Press,, pp. 101-109; — and Singh, S. R.: 1977, Orthogonal Polynomials and Padé Approximants associated with a system of arcs, *J. Approx. Theory* **21**, pp. 1-42.
23. Stahl, H.: 1997, The convergence of diagonal Padé approximants and the Padé conjecture, *J. Comp. Appl. Math.* **86**, pp. 287-296.
24. Lubinsky, D. S.: 1985, Padé Tables of Entire Functions of Very Slow Growth, *Const. Approx.* **1**, pp. 349-358; — : 1988, Padé Tables of Entire Functions of Very Slow Growth II, *Const. Approx.* **4**, pp. 321-339.
25. de Montessus de Ballore, R.: 1902, Sur les fractions continues algébriques, *Bull. Math. de France* **30**, pp. 28-36.
26. Beardon, A. F.: 1968, On the location of poles of Padé approximants, *J. Math. Anal. Appl.* **21**, pp. 467-474.
27. Lubinsky, D. S.: 1987, Uniform Convergence of Rows of the Padé Table for Functions with Smooth Maclaurin Series Coefficients, *Const. Approx.* **3**, pp. 307-330.
28. Wilson, R.: 1930, Divergent Continued Fractions and non-polar singularities, *Proc. London Math. Soc.* **30**, pp. 38-57.
29. Baker, G. A., Jr.: 1975, *Essentials of Padé Approximants*, Academic Press, New York.

30. Walin H.: 1974, The convergence of Padé approximants and the size of power series coefficients, *Appl. Anal.* **4**, pp. 235-251.
31. Fatou, P.:1906, Série trigonométrique et série de Taylor, *Acta Math.* **30**. pp. 335-400.
32. Dienes, P., *The Taylor Series*, Dover, New York, 1957; Theorem 105.IV.
33. Vitali, G.: 1903, Sopra le serie di funzioni analitiche, *Rend. R. Inst. Lombardo* (2) **36**, pp. 772-774.
34. Leau, L.:1899, Recherches sus les singularités d'une fonction définie par un développement de Taylor, *J. de Math.* (5) **5**, pp. 365-425.
35. Faber, G.: 1903, Über die Fortsetzbarkeit gewisser Taylorschen Reihen, *Math. Annalen* **57**, pp. 369-388.
36. Wigert, S.: 1900, Sur les fonctions entières, *Obversigt af Svenska Vetenskaps-Akad. Forhandl.* **57**, pp. 1001-1011.